A Dissection of the Square into Similar Right Triangles.

A.J. Lee
(unpublished typescript from 3 March 1999)

The type of dissection considered here is based on the following construction (fig. 1). Two intersecting straight lines K, L make an angle θ<90° at their point of intersection. From a point A on one of the lines drop a perpendicular to the other line, meeting it at B. From B drop a perpendicular back on to the first line, to point C. Continue this process, alternately dropping perpendiculars between the two lines, to produce a pyramid of diminishing right triangles which become indefinitely small as they approach the point of intersection of lines K, L. It is easy to see that all triangles are similar, and that each has angles opposite the right angle equal to θ and 90°- θ.

In the series of perpendiculars (each of which constitutes the longer base or height of one triangle and the hypotenuse of the next), if a given perpendicular has length h, say, then successively smaller perpendiculars have lengths $h.\cos\theta$, $h.\cos^2\theta$, $h.\cos^3\theta$... $h.\cos^{n-1}\theta$, $h.\cos^n\theta$. The size ratio of larger to next smaller triangles is $1/\cos\theta$.

The lengths of the segments marked out on the initial lines K, L by successive perpendiculars have a size ratio of $1/\cos^2\theta$, since those along either line belong to every other triangle.

Choose any triangle, e.g. triangle ABC in fig. 1, and rotate it through 90° about point B so that its hypotenuse coincides with line L (fig. 2). Label this new triangle BDE, so that D is the right angle, and E lies on line L. In the example illustrated, side DE is collinear with one of the perpendiculars between lines K, L, and we can extract square $CBDF$ (fig. 3) which shows a dissection into seven similar right-angled triangles. For arbitrarily chosen θ sides DE and EF will not necessarily be collinear. The condition that ensures their collinearity is that the sum of the segment lengths $h.\sin\theta.\cos\theta$, $h.\sin\theta.\cos^3\theta$ and $h.\sin\theta.\cos^5\theta$ should equal the side length of the square, i.e.

$$\sin\theta(\cos\theta+\cos^3\theta+\cos^5\theta) = 1 \qquad (1).$$

Angle θ can be determined by trial and error using a hand calculator, in which case an easier expression can be based on the sum of lines DE and EF:

$$\tan\theta+\cos^6\theta = 1 \qquad (2).$$

If the number of triangles inside the square is n, these expressions generalise as

$$\sin\theta(\cos\theta+\cos^3\theta+\cos^5\theta+...+\cos^{n-2}\theta) = 1 \qquad (3)$$

$$\tan\theta+\cos^{n-1}\theta = 1 \qquad (4).$$

All the dissections mentioned here follow the same pattern as that in fig. 3, i.e. each consists of a sequence of an even number of triangles sitting on the hypotenuse of the single, largest triangle. The number n of right triangles in each dissection is thus always odd.

The distribution of integral solutions (and therefore possible dissections) can be shown graphically, as in figs. 4 and 5. Equation (4) has no integral solutions for $n<7$, but for $n \geq 7$ there are two topologically identical solutions for each integral value of n, differing only in the size of angle θ. If we put $s = \tan\theta + \cos^{n-1}\theta$ then fig. 4 shows the curve for which $s = 1$. This separates areas in the plane where s is greater or less than 1. The curve has asymptotes at $\tan\theta = 0$ and $\tan\theta = 1$. An alternative way of visualizing the location of integral solutions is to represent the *surface* defined by $s = \tan\theta + \cos^{n-1}\theta$ in (s, $\tan\theta$, n)-space. Fig. 5 shows the surface collapsed on to the (s, $\tan\theta$)-plane with superimposed cross sections at $n = 1$ to 15, 51, 101 and 1001. Integral solutions relevant to the dissections under discussion occur where these sections intersect the line $s = 1$, or alternatively where the surface intersects the plane $s = 1$. Fig. 5 clearly shows that cross sections of the surface at $n = 1, 3, 5$ all lie above $s = 1$, and therefore no solutions are possible below $n = 7$.

Finally, the appearance of the first few pairs of dissections is given in fig. 6. Note that when $\theta = 45°$ the dissection contains an infinite sequence of 45° right triangles. If $\theta = 0°$ the square becomes filled with an infinite number of "triangles" of zero area.

Retyped October, 2009.

Publication of the following three relevant papers is acknowledged:

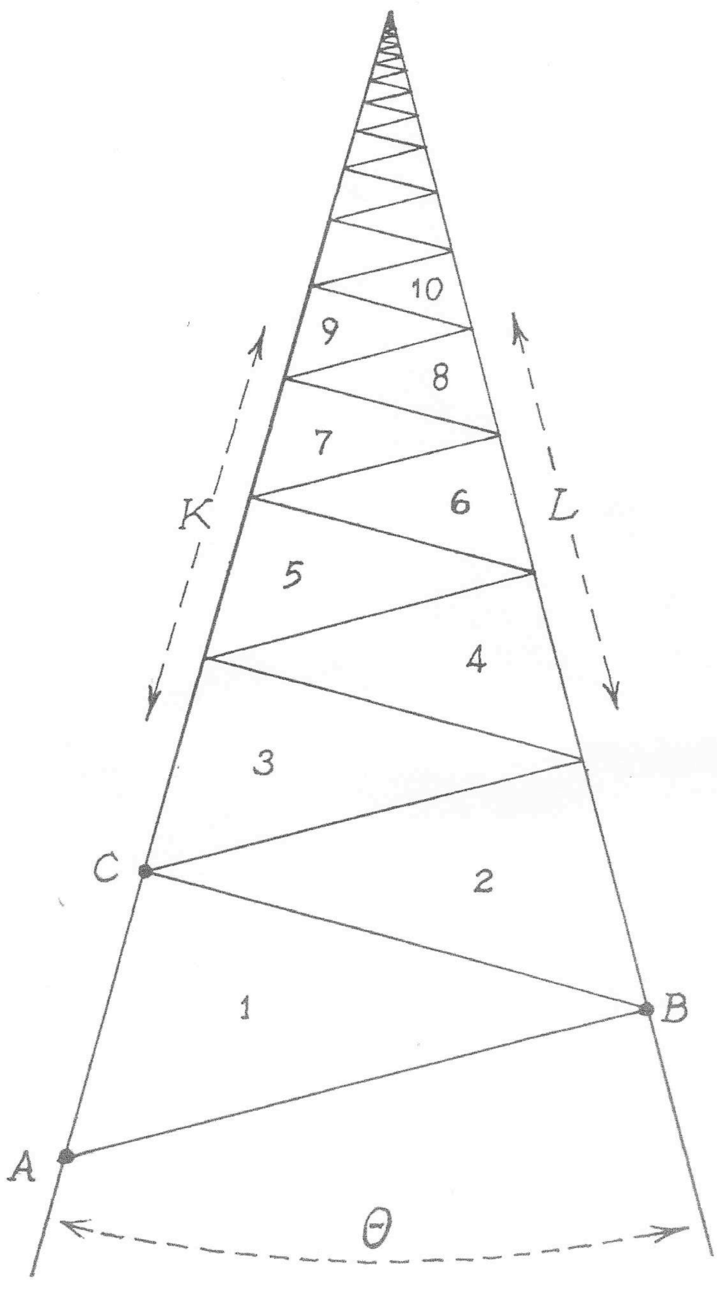
(1) M. Laczkovich (1990). Tilings of polygons with similar triangles. *Combinatorica* **10** (3) 281-306.

(2) M. Laczkovich and G. Szekeres (1995). Tilings of the Square with Similar Rectangles. *Discrete and Computational Geometry* **13**: 569-572.

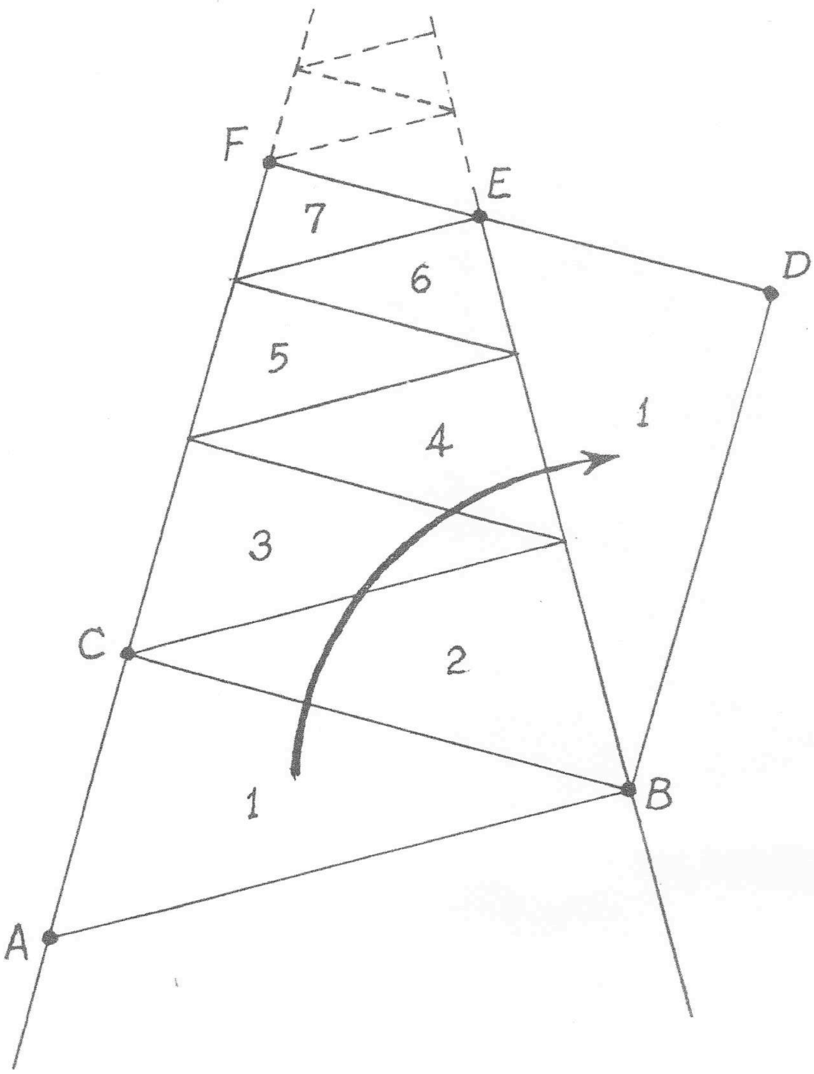
(3) Balázs Szegedy (2001). Tilings of the square with similar right triangles. *Combinatorica* **21** (1) 139-144.

I note that Laczkovich & Szekeres (2) give a drawing (p. 572) of one of the two solutions for $n = 7$ offered in the present unpublished typescript.

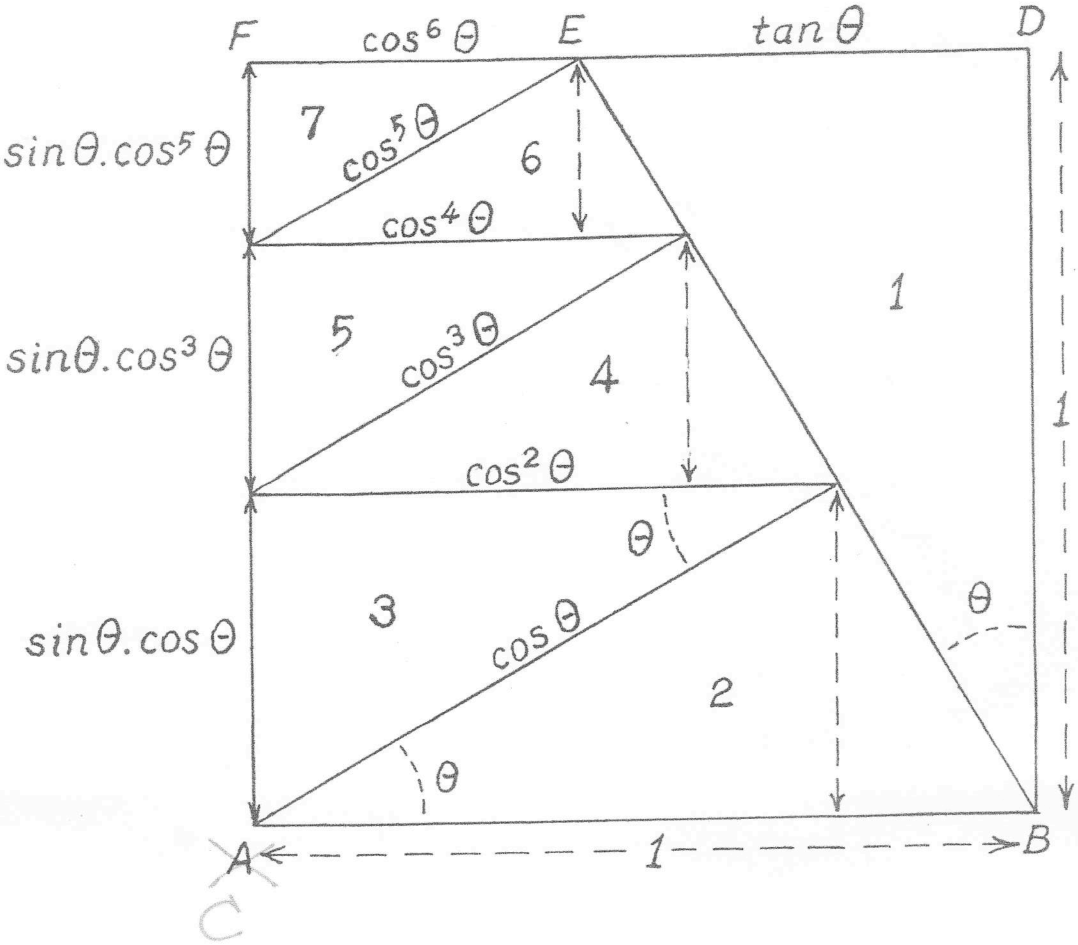
A. J. Lee "A dissection..." FIG. 1



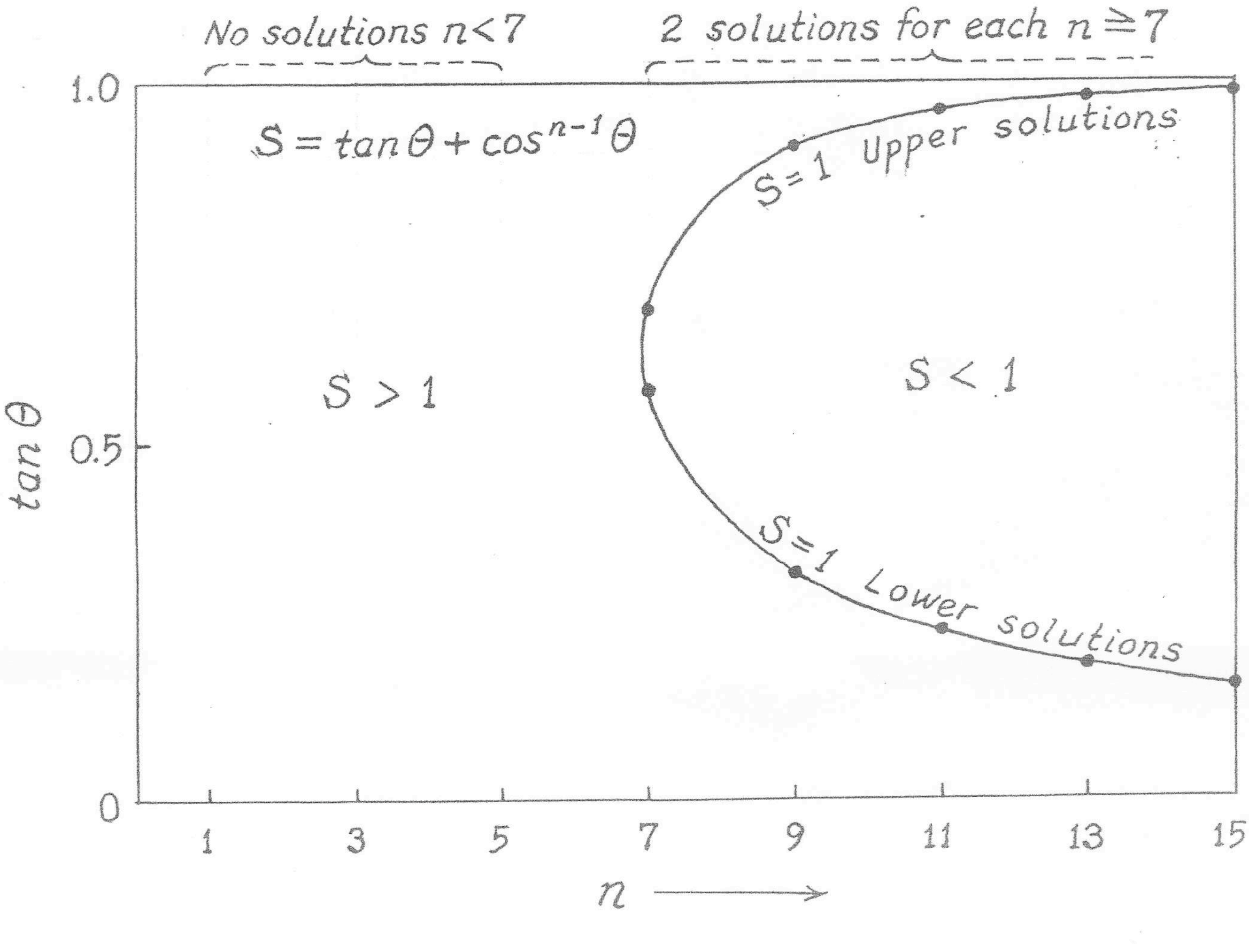
A. J. Lee "A dissection..." FIG. 2



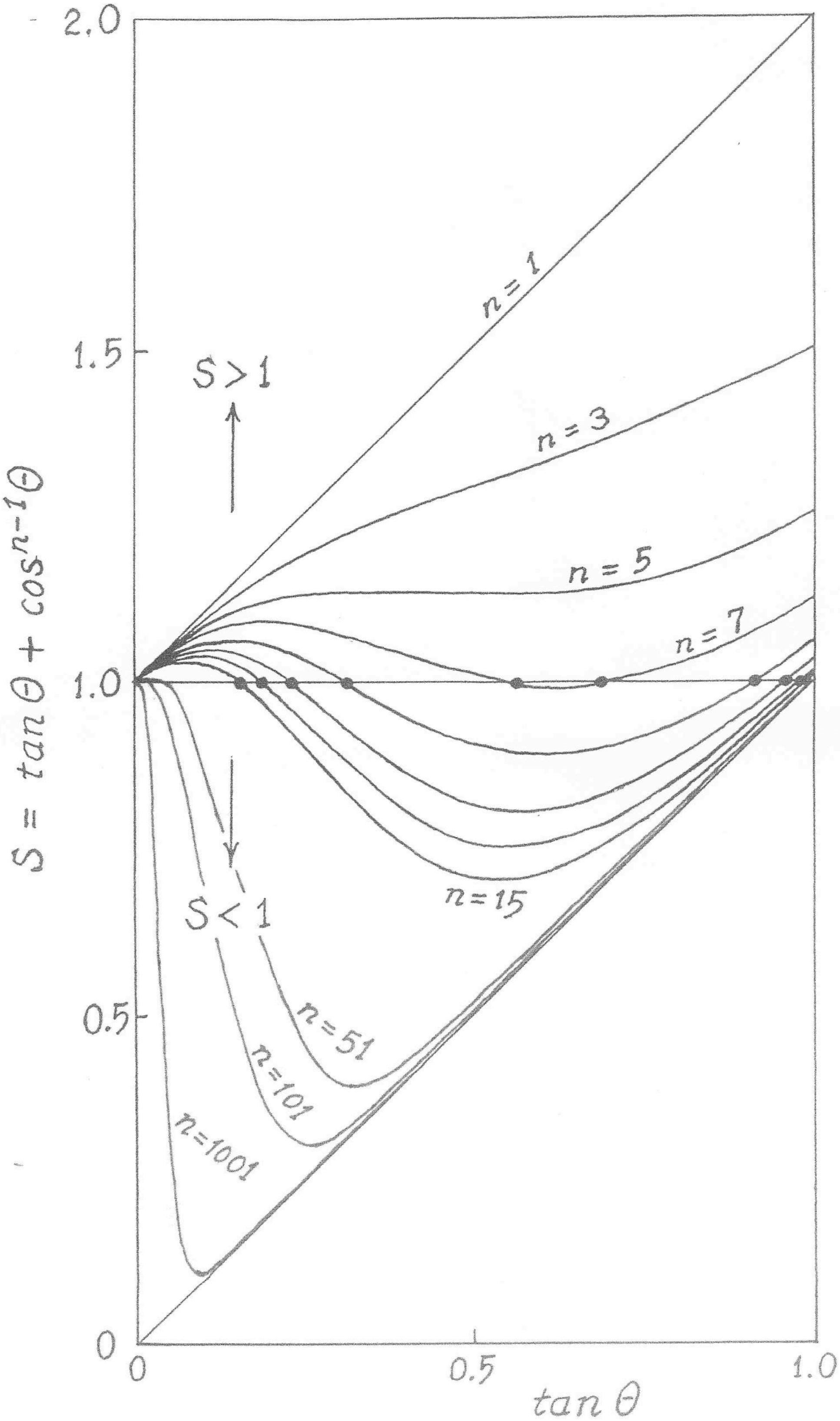
A. J. Lee "A dissection..." FIG. 3

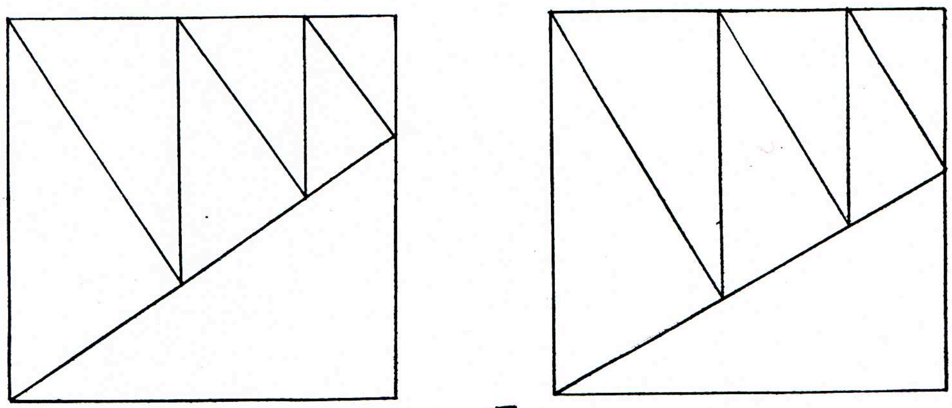


A.J. Lee "A dissection..." FIG. 4

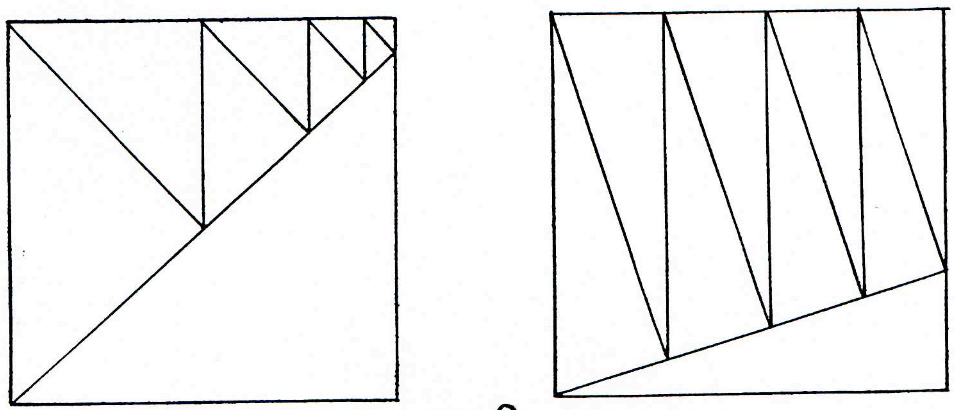


A. J. Lee "A dissection..." FIG. 5

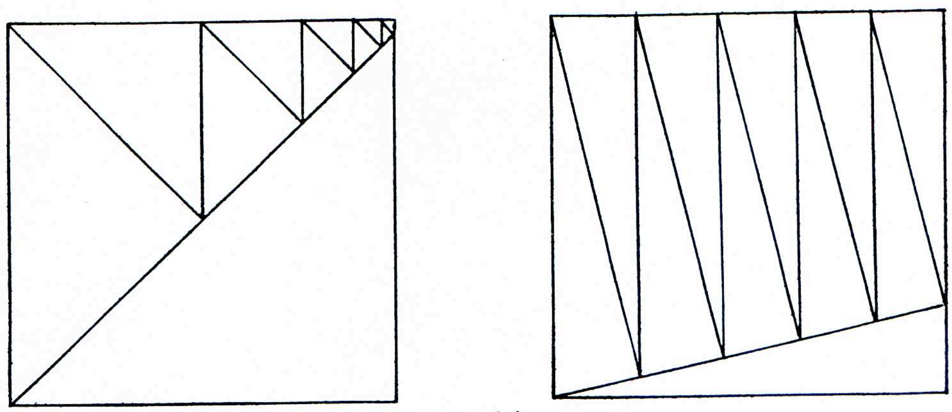




$n = 7$



$n = 9$



$n = 11$

Fig.